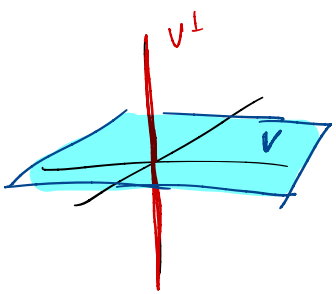
Def: Let V be a subspace of $\mathbb{R}^n$.
The orthogonal complement of V is

$$V^\perp = \{ w \in \mathbb{R}^n \mid w \cdot v = 0 \ \forall \ v \in V \}$$

(set of vectors orthogonal to all vectors in V)

Ex: Let $V = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \in \mathbb{R}^3 \mid x, y \in \mathbb{R} \right\}$.

Then $V^\perp = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \in \mathbb{R}^3 \mid z \in \mathbb{R} \right\}$



Key Feature: Let V be a subspace of $\mathbb{R}^n$.
Then $V^\perp$ is also a subspace of $\mathbb{R}^n$

Proof:

- Since $\vec{0} \cdot v = 0 \ \forall \ v \in V$, we have $\vec{0} \in V^\perp$.
- Let $w_1, w_2 \in V^\perp$. Then $w_1 \cdot v = w_2 \cdot v = 0 \ \forall \ v \in V$.

Hence $(w_1+w_2)\cdot v = w_1\cdot v + w_2\cdot v = 0+0=0 \quad \forall v\in V$

Thus $w_1+w_2 \in V^\perp$.

- Let $w\in V^\perp$ and $c\in\mathbb{R}$.
  Then $(cw)\cdot v = c(w\cdot v) = c\cdot 0 = 0 \quad \forall\, v\in V$
  $\Rightarrow cw\in V^\perp$. ∎

Properties: Let V be a subspace of $\mathbb{R}^n$.

a) $(V^\perp)^\perp = V$

b) $V\cap V^\perp = \{\vec{0}\}$

c) $\dim V + \dim V^\perp = n$

Proof: a) By definition (check)

b) On next HW

c) Uses orthogonality of four subspaces below (recitation)

The Four Subspaces, Revisited

Recall: if A is $m\times n$, then $C(A), N(A^T)\subseteq\mathbb{R}^m$ and $C(A^T), N(A)\subseteq\mathbb{R}^n$

It turns out they are orthogonal complements! That is:

- $N(A)^\perp = C(A^T)$ (and $N(A) = C(A^T)^\perp$)
- $C(A)^\perp = N(A^T)$ (and $C(A) = N(A^T)^\perp$)

Proof: Note the second follows from the first by setting A to A^T. So we only need to prove the first.

Step 1: Show $C(A^T) \perp N(A)$ (this implies $C(A^T) \subseteq N(A)^\perp$)

Let r_i^T denote the ith row of A (so $r_i \in C(A^T)$)

Let $v \in N(A)$ be arbitrary. Then $\vec{0} = Av = \begin{bmatrix} r_1 \cdot v \\ r_2 \cdot v \\ \vdots \\ r_m \cdot v \end{bmatrix}$

Thus $r_i \cdot v = 0 \quad \forall\, i$.

Let $w \in C(A^T)$. Then $w = x_1 r_1 + \cdots + x_n r_n$ for some weights $x_1, \ldots, x_n$

Then $v \cdot w = v \cdot (x_1 r_1 + \cdots + x_n r_n) = x_1(v \cdot r_1) + \cdots + x_n(v \cdot r_n) = x_1(0) + \cdots + x_n(0) = 0$

So $v \cdot w = 0 \quad \forall\, v \in N(A)$ and $w \in C(A^T) \Rightarrow C(A^T) \perp N(A)$

Step 2: Show if w is orthogonal to $N(A)$, then $w \in C(A^T)$
(this implies $C(A^T) \supseteq N(A)^\perp$)

Let $w \cdot v = 0 \quad \forall \quad v \in N(A)$.

If $w \notin C(A^T)$, let $A' = \begin{bmatrix} A \\ w^T \end{bmatrix}$. Note A' is $(m+1) \times n$.

Then $N(A) = N(A')$ but $C(A')^T \supsetneq C(A^T)$ (A' has a bigger row space than A)

$\Rightarrow \dim N(A') = \dim N(A)$ and $\text{rank}(A') > \text{rank}(A)$

$\Rightarrow \dim N(A') + \text{rank}(A') > \dim N(A) + \text{rank}(A) = n$ by rank-nullity thm.

But A' only has n columns, so we've contradicted the rank-nullity theorem applied to A'. Hence $w \in C(A^T)$.

Since $C(A^T) \supseteq N(A)^\perp$ and $C(A^T) \subseteq N(A)^\perp$, we have $C(A^T) = N(A)^\perp$ ∎

Key Point: Any subspace V is the column space of a matrix A. Hence $V^\perp = C(A)^\perp = N(A^T)$

So we have a systematic way to find orthogonal complements. (See example in next recitation)

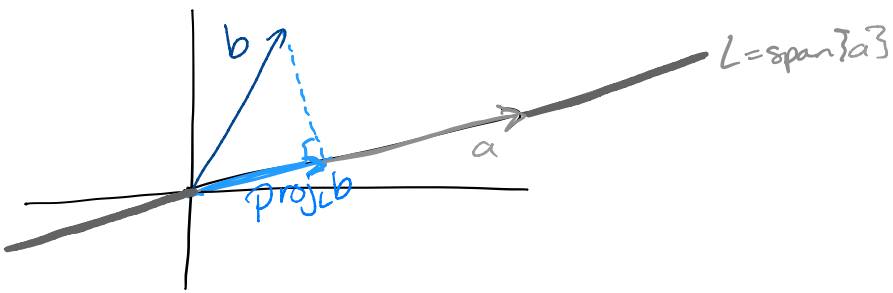
(Columns are basis vectors)

Projections

Let $a \in \mathbb{R}^n$ and $L = \text{Span}\{a\}$.

Let $b \in \mathbb{R}^n$.

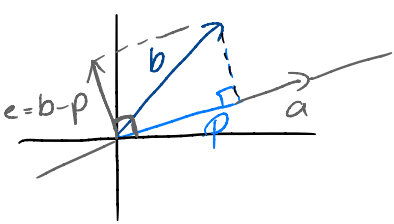
The projection of b onto L is the vector p in L that is closest to b (Notation: $p = \text{proj}_L b$ or $\text{proj}_a b$)



Since $p \in \text{Span}\{a\}$, $p = \hat{x}a$ for some scalar $\hat{x}$. What is $\hat{x}$?

Consider the right triangle

The vector $e = b - p$ ("error") is perpendicular to a.



Hence $a \cdot (b - p) = 0$

$$a \cdot b = a \cdot (\hat{x}a)$$

$$a \cdot b = \hat{x}(a \cdot a)$$

$$\Rightarrow \hat{x} = \frac{a \cdot b}{a \cdot a}$$

Thus the projection of b onto $L = \text{span}\{a\}$ is the vector

$$\text{proj}_L b = \frac{a \cdot b}{a \cdot a} a$$